Some Properties of the Fractional Equation of Continuity and the Fractional Diffusion Equation

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Anomalous phenomena are often considered to be relate to complex structures of the materials. The dispersive current observed in some amorphous semiconductors is interpreted by the fractal structures of the potential. The frequency dependence of the response of some viscoelastic materials is attributed to the complex structures of high polymers. These phenomena can be explained by fractional derivative models (e.g. [1]). The pth order fractional derivative is defined by

$$\frac{d^p f(t)}{dt^p} = \frac{d^n}{dt^n} \int_a^t \frac{(t-\tau)^{n-p-1}}{\Gamma(n-p)} f(\tau) d\tau, \qquad (1)$$

where the integer n satisfies $n-1 \le p < n$.

In some complex structure, the equation of continuity can be represented by the fractional equation of continuity (FEC) defined by

$$\frac{\partial^p \rho}{\partial t^p} + \nabla \cdot \mathbf{J} = \frac{\partial^p \mu}{\partial t^p}, \qquad (2)$$

where $0 < p < 1$. The RHS of (2) is the source term, in which $\partial \mu(x,t)/\partial t$ is the source in the usual sense: the mass creation rate per unit volume per unit time interval. The form of eq. (2) assures the conservation of mass in the sense that the total mass in the volume in consideration is the sum of the initial mass and the created mass by the source term [2].

In the FEC, the increment of the density in a volume element depends on the memory of $(\rho - \mu)$ as well as the present $\nabla \cdot \mathbf{J}$ and $\partial\mu/\partial t$ because of the integrated form of the fractional derivative. The memory of $(\rho-\mu)$ can be replaced by the memory of $\nabla \cdot \mathbf{J}$ using eq. (2) (see [2] for detailed explanation). The memory of $\nabla \cdot \mathbf{J}$ works in the opposite sense to the usual $\nabla \cdot \mathbf{J}$. Thus, if $\nabla \cdot \mathbf{J}$ works to increase the density at $\tau(< t)$, then it works to decreases the density at the time t.

An example of the curious behavior of the FEC is presented by the fractional diffusion equation (FDE). The FDE is derived from eq. (2) by substituting $\mathbf{J} = -K\nabla\rho$, where $K > 0$ is the (fractional) diffusion coefficient.

Figure 1 shows a solution of the initial and the boundary value problem of FDE for the spherical symmetric case. In the central region the sink of the form, $\partial\mu/\partial t = -\alpha(r)\rho$, is assumed for $r < 10$. A ring of dip forms around the central peak. This structure is explained as follows. In the early stage, the absorption is not strong enough to make a dip at the center. Then, the term $K\nabla^2\rho$ is the negative sign, which works to enhance the central peak afterward as the memory effect.

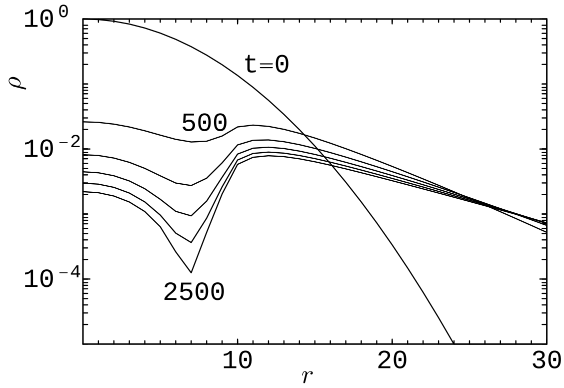

Figure 1: A solution of the FDE with the sink at the central region, $\alpha = 0.01$ in $0 \le r < 10$ and $\alpha = 0$ in $r \ge 10$. The other parameters for the FDE are, $p = 1/2$ and $K = 1$. The curves show the densities at $t = 0, 500, 1000, \ldots, 2500$.

In the outer region, the curvature of ρ is small. Thus, it is easy to have positive sign of $K\nabla^2\rho$ by absorption. Once a dip forms the memory of the positive sign of $K\nabla^2\rho$ tends to deepen the minimum ρ. The profile of the central region is sensitive to the initial distribution and the form of $\alpha(r)$. If the central absorption region is narrow, a dip forms at the center.

References

[1] M. Fukunaga, The 1st IFAC Workshop on Fractional Differentiation and its Applications, ENSEIRB, Bordeaux, France, July 19-21 (2004) p. 558, p. 564.

[2] M. Fukunaga, Intern. J. Appl. Math. **14** (2003) p. 269.