Investigation on Network Structure Formed by Traces of Random Walkers[1]

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Summary

We have investigated a network structure formed by traces of random walkers from viewpoints of spatial distribution of vertex degree and correlation between two adjacent vertices. We obtained positive correlation for degree-degree correlation and cluster-cluster correlation. On the other hand, degree-cluster correlation can be positive or negative according to the situation. It should be noticed that such signs of correlation can be interpreted by the formulation process of the network and degree-cluster correlation is related to the capacity for vertices to oppose aging of edges.

1. Introduction

The importance of network structures has been recognized recently in diverse research fields. One reason is due to the discovery of common structures typically known as scale free or small world. These properties can be found in the real world, such as internet, social and neural network, and metabolic systems[1]. Mathematical modeling of complex networks by using graph theory is useful for understanding how such common structures are generated. In studies of complex networks, it was found that simple principles can sometimes provide important topological properties of real systems. For example, scale free model (BA model) and small world model (WS model) are famous and epoch-making models[2, 3].

We have proposed a network evolution model considering a physical configuration and transports between the elements of system [4]. The consideration for physical configuration in this model is that the network starts from simple lattices such as one-dimensional or two-dimensional square lattice. This initial lattice means a simple geographical relation between the vertices. The transports are represented abstractly as random walkers. The important assumption for the walkers is that movement of random walkers causes a new creation of bypass between nodes and strengthens the links where the walker passes, while links which are rarely used tend to be extinct except links on the initial lattice. In other words, transports between nodes abstracted as random walks determine the rise and fall of connections in the network. The remarkable characteristics of this model are that the typical structure of complex network such as "appearance of a few vertices with large degree", "large clustering coefficient", and "small mean distance between two vertices" can be simultaneously observed by adjusting parameter values.

In this report, we will focus on other aspects of network topology, such as spatial distribution of vertex degree, and correlation between degrees on two adjacent vertices. The spatial distribution of vertex degree is interesting because our model sometimes provide complicated form of networks on the original lattice, which are related to a diffusion process of random walkers. Concept of correlation between degrees on two adjacent vertices have drawn considerable attention recently [5, 6], because positive and negative correlation have been observed according to the kinds of real networks under consideration. For example it is known that positive degree correlation is obtained for social networks, and negative degree correlation is obtained for internet, neural network, etc. Our calculation yields positive degree correlation for our model.

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The rules for the network evolution in our model are described as follows. Let take w random walkers in two-dimensinal square lattice, which can move randomly and independently between joined two vertices. We assumed in this paper all random walkers start from the same vertex at initial time. Edges are created between a vertex where a walker currently stays and vertices where the walker stayed one step before and two steps before. But, if there already exists these edges, add one on the number "strength of the edge" which is defined on each edge. Edges of strength 1 will be extinct with probability p_d per unit time, and edges with strength larger than 1 will lose their strength by one with probability p_d per unit time, although edges joining two neighboring vertices in the original two-dimensional lattice can never be extinct. Notice that the resultant network structure formed by these rules are different according to change of parameter w and p_d.

2. Spatial Distribution of Vertex Degree

In this section, we present some examples of spatial distribution of vertex degree. They provide intuitive understandings of the problem. All random walkers are forced to start from the same vertex in the calculations in order to investigate interaction between movements of different random walkers. Figure 1 presents a case for $p_d = 0$. In this case, we can find a very clear power law of frequency distribution of vertex degree [7]. The figure shows that vertices with large degree are concentrated on a area where the walkers started at initial time. In our model all the random walkers leave many edges incident to vertices where they have passed. As a consequence, the formulated network is determined by statistics of multiple random events although we observed only an example of movement of random walkers. For brevity, let us call the shape of spatial distribution of vertex degree as "*degree island*".

Figure 2 presents a degree island for the case of$p_d = 0.0004$. It is found that the height of the degree island becomes lower than that for the case of $p_d = 0$. But, we have shown in our previous work [7] that the power law of frequency distribution of vertex degree maintains except the tail of the distribution owing to the smallness of p_d. It is observed that a small island is divided from the center of the large island. But such division is considered as a rare event and the most of

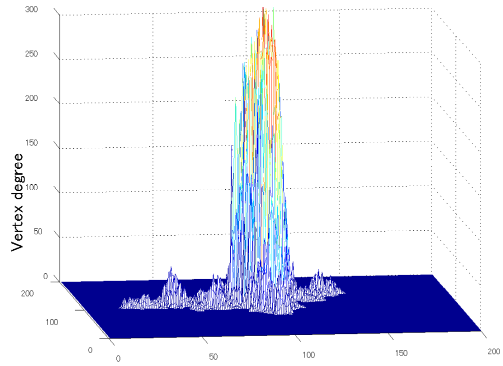

Fig.1 Spatial distribution of vertex degree for the case of $p_d = 0$ and $w = 5$. Five random walkers started from the same coordinate of $(150, 150)$.

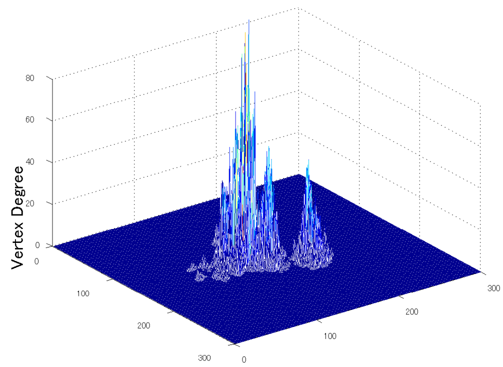

Fig.2 Spatial distribution of vertex degree for the case that p_d is small ($p_d = 0.0004$, $w = 5$).

random walkers can hardly escape from the degree island, for many peaks of vertex degree remains to stay in the center of the degree island, which attracts the random walkers by the many edges incident to the vertices with large degree.

Finally, Figure 3 presents a case for $p_d = 0.004$ and $w = 1$. In this case, owing to the small capacity of new creation of edges, vertex with large degree can not survive although the extinction probability of edges is the same as the above case. As a result, frequency distribution of vertex degree exhibits a exponential decay [7]. Frequent extinction of edges allows an emigration of the formulated network. The degree island does not have a stable shape that occupies the same position.

3. Correlation between Two Adjacent Vertices

In this section, the change of network described in the previous section is investigated by calculation of correlation between two adjacent vertices. We have

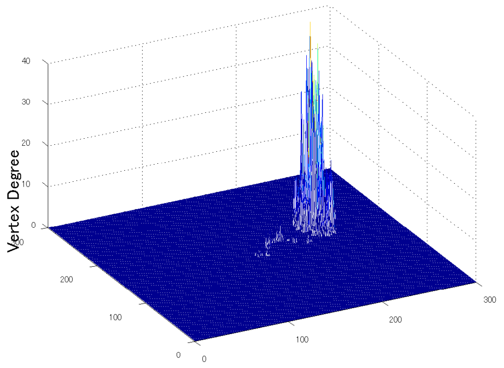

Fig.3 Spatial distribution of vertex degree for the case that $p_d = 0.004$ and $w = 1$. Random walkers started from the coordinate of $(150, 150)$, where there no longer exists a network which was formulated during early times.

calculated three kinds of correlation, that is, degree-degree (d-d) correlation, cluster-cluster (c-c) correlation, and degree-cluster (d-c) correlation between two adjacent vertices. Notice that Cluster strength per a vertex "k" is defined as the next formula,

$$\sum_{i,j \in s(k)} \chi_{ij} / \sum_{i,j \in s(k)} 1, \qquad (1)$$

where $s(k)$ is a set of vertices joined to a vertex "k", and χ_{ij} is a function which takes only values 1 or 0 according to whether or not the vertices i and j are joined. Figure 4 and 5 present examples of scatter diagram of the d-d correlation and the c-c correlation, respectively.

As expected from the scatter diagrams, our calculations have yielded positive correlation coefficients for the d-d correlation and the c-c correlation. Such positive correlation can be interpreted intuitively from the movement of random walkers, for random walkers tend to float around vertices with large degree. According to the rule for new creation of edges, the movement of a random walker from vertex a and c via vertex b creates a new link between vertex a and c. It is natural to expect that such new creation frequently occurs between vertices with large degree, but rarely happens between vertices with small degree, because of the tendency that random walkers float around vertices with large degree.

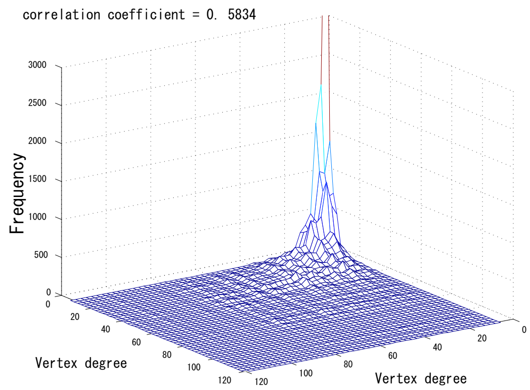

Fig.4 Scatter diagram of the d-d correlation. The correlation coefficient is calculated as 0.5834.

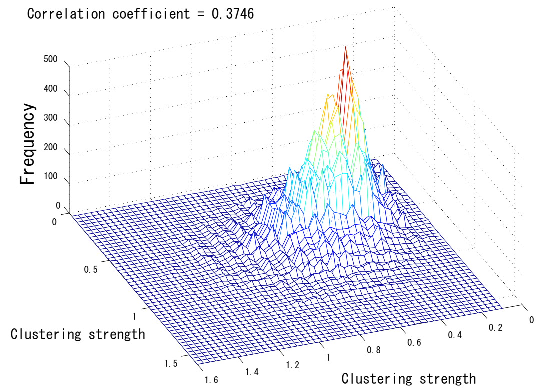

Fig.5 Scatter diagram of the c-c correlation. The correlation coefficient is calculated as 0.3746.

The more interesting results are obtained for the values of d-c correlation, for d-c correlation is found to be able to take both positive and negative values according to the change of p_d. Figures 6 and 7 show a change of scatter diagrams with p_d.

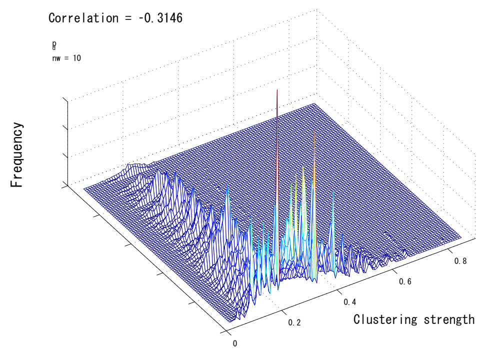

Fig.6 Scatter diagrams obtained for $p_d = 0.0004$ and $w = 10$. The correlation coefficient is calculated as -0.3146.

Comparing Figure 6 with 7, we can find a shift in the clustering strength for vertices with large degree. In Figure 6, vertices with large degree keeps adjacent vertices with highly clustered vertices. As a

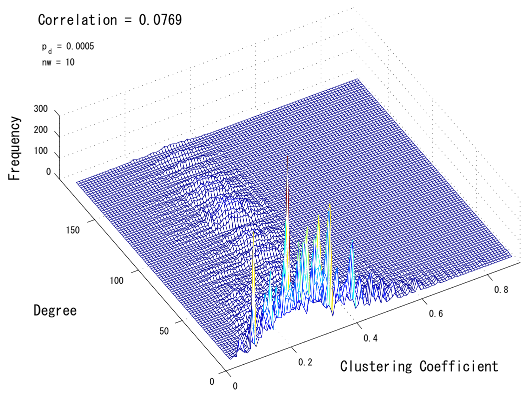

Fig.7 Scatter diagrams obtained for $p_d = 0.0005$ and $w = 10$. The correlation coefficient is calculated as 0.0769.

result, negative correlation is obtained. This behavior of negative correlation is sometimes observed when p_d is small but not 0. As shown in Figure 7, negative correlation becomes suddenly positive as the extinction probability p_d is over a certain value. It should be noted that the largest vertex degree is also reduced as p_d is over a certain value.

4. Interpretation of Negative Correlation Coefficient

It should be noted that the clustering strength per a vertex determines the capacity of new creation of links around the vertex. Suppose that a random walker moves in the order vertex 1, vertex 2, and vertex 3 (see Figure 8). Then, whether vertex 1 and 3 are newly linked is determined by whether vertex 1 and 3 have already been linked or not. The probability of existence of a link between 1 and 3 is equal to the clustering strength around vertex 2.

Furthermore, this property of clustering strength yields the fact that a vertex whose adjacent vertices are low clustered has an advantage to gain new links. Suppose that a random walker moves in the order vertex 1 and 2. Since the walker will go to arbitrary vertices joined to vertex 2, the probability of a new creation of edge incident to vertex 1 is given by the formula

$$1 - L_{(1\to s(2))}/(L_{s(2)} - 1), \tag{2}$$

where $L_{(1\to s(2))}$ means number of edges between vertex 1 and all the vertices joined to vertex 2, and $L_{s(2)}$ means number of vertices joined to vertex 2. The second term of probability (2) is roughly estimated as clustering strength of vertex 2 if the degree of vertex 2 is large.

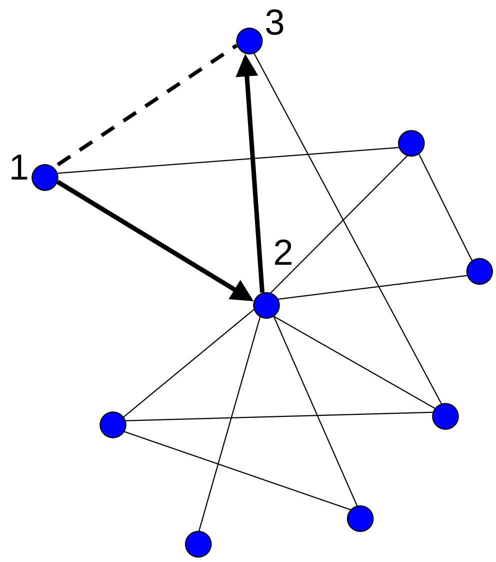

Fig.8 If a random walker moves in the order vertex 1 and 2, then the walker will go to arbitrary vertices joined to vertex 2. (Vertex 3 is accidentally selected in this figure.) Whether a edge incident to vertex 1 is created or not is dependent on the clustering strength around vertex 2.

This relation between clustering strength and capacity of new creation of edges helps understanding of the meaning of the negative d-c correlation observed in cases that p_d is small. Maintenance of vertices with large degree must need adjacent low clustered vertices. Otherwise, continuous extinction of edges overcoming new creation of edges reduces the degree of the vertex. In other words, vertices with large degree and with adjacent highly clustered vertices tend to be eliminated.

5. Concluding Remarks

We have investigated a network structure formed by traces of random walkers from viewpoints of spatial distribution of vertex degree and correlation between two adjacent vertices. Observation of spatial distribution of vertex degree is useful to capture network formulation processes such as localized stationary structure, collapse of the network, and emigration of the network. However, the problem of how to investigate the complex shapes of the degree island is remained to be solved.

According to our calculation, degree-degree correlation and cluster-cluster correlation tend to have positive correlation coefficients. This positive correlation is probably an essential property of the network formed by traces of random walkers, because such positive correlation is considered as a result of movements of random walkers, which tend to be stay at vertices with large degree. It is important to recognize that a particular formulation process of the network determines whether the correlation is positive or negative.

We also pointed out that degree-cluster correlation should be related to the capacity for vertices to oppose aging of edges. Negative degree-cluster correlation means that vertices with large degree tend to have adjacent low clustered vertices. According to our interpretation, such situation have an advantage of maintaining vertices with large degree opposing continuous extinction of edges. It is interesting that the network can take positive or negative degree-cluster correlation according to the extinction probability of edges.

It should be noticed that scatter diagrams does not always indicate a linear relation between two variables. This fact means that calculation of correlation coefficients is not sufficient for analyzing the relation between two variables. Although we have calculated only correlation coefficients so far, whether the two variables have linear relation or not should be considered, and appropriate statistical consideration should be introduced if needed.

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