A Partial Differential Equation with respect to States as the Limit of the Moving Particle Semi-implicit scheme

In Computational Fluid Dynamics

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In computational fluid dynamics, to simulate free surface motion such as splashing water, breaking waves and so on, the Moving Particle Semi-implicit method (MPS, Moving Particle Simulation method) is a powerful numerical analysis theory [Koshizuka and Oka 1996]. This paper seeks the mathematical foundation of MPS method. For example, in the Moving Particle Semi-implicit method (MPS, Moving Particle Simulation method) is a powerful numerical analysis theory [Koshizuka and Oka 1996]. This paper seeks the mathematical foundation of MPS

Let \( x = (x_1, x_2, \cdots, x_N) \) be a space point (an Eulerian coordinate) in Euclidean space \( \mathbb{R}^N \) where fluid flow deforms. Let \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbb{R}^N \) be an initial position of a fluid particle at initial time \( t = 0 \). In these sense, \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \) denotes each fluid particle (a Lagrangian coordinate). \( D/Dt \) denotes a material derivative (Lagrangian derivative) with respect to time \( t \).

For \( l \) th particle \( \alpha_{[l]} \), let \( u(t, \alpha_{[l]}) \) be its position and let \( v(t, \alpha_{[l]}) \) be its velocity at any time \( t \geq 0 \). We formulate Navier-Stokes equation in MPS by

\[
\frac{D}{Dt} v(t, \alpha_{[l]}) - \frac{\mu}{\rho(t, \alpha_{[l]})} \nabla \cdot \nabla v(t, \alpha_{[l]}) + \frac{1}{\rho(t, \alpha_{[l]})} \nabla P(t, \alpha_{[l]}) = f(t, \alpha_{[l]}) \tag{1}
\]

where the Laplacian \( \nabla \cdot \nabla \) and the gradient \( \nabla \) are defined by suitable interactions of the \( M \) particles \( \alpha_{[l]} \) \( (l = 1, 2, \cdots, M) \) [Koshizuka and Oka 1996].

As the number \( M \) of the fluid particles increases, the MPS scheme (1) converges to a partial differential equation

\[
\frac{D}{Dt} v(t, \alpha) - \frac{\mu}{\rho(t, \alpha)} \sum_{i=1}^{N} \frac{\partial^2 v(t, \alpha)}{\partial u_i(t, \alpha)^2} + \frac{1}{\rho(t, \alpha)} \left( \frac{\partial P(t, \alpha)}{\partial u_i(t, \alpha)} \right) \delta_1 \delta_2 \cdots \delta_N = f(t, \alpha) \tag{2}
\]

This equation (2) is neither Lagrangian form nor Eulerian form. It is a hybrid of Lagrangian form and Eulerian form. Since this partial differential equation contains partial derivatives with respect to a state \( u \), it is not a familiar partial differential equation who has only partial derivatives with respect to a space point \( x \).

We assume that the fluid flow is incompressible and let \( \rho_0 \) be a constant value which expresses \( \rho(t, \alpha) \). To analyse the partial differential equation (2), we can eliminate \( \rho(t, \alpha) \) by

\[
\frac{1}{\rho(t, \alpha)} = \frac{1}{\rho_0} \det \left( \frac{\partial u_i(t, \alpha)}{\partial \alpha} \right) \tag{3}
\]

and we can obtain a partial differential equation whose unknowns are the position \( u(t, \alpha) \), the velocity \( v(t, \alpha) \), and the pressure \( P(t, \alpha) \).

Based on the fluid’s incompressibility, we obtain that the volume expansion des \( \partial u/\partial \alpha \) becomes a constant value 1, i.e.

\[
\det \left( \frac{\partial u(t, \alpha)}{\partial \alpha} \right) = 1 \tag{4}
\]

for any Lagrangian coordinate \( \alpha \) and any time \( t \). By the chain law

\[
\frac{\partial}{\partial u_i(t, \alpha)} = \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial u_i(t, \alpha)} \frac{\partial \alpha_j}{\partial \alpha} \tag{5}
\]

we can transform the limit equation (2) of MPS scheme into the equation who has only partial derivatives with respect to the Lagrangian coordinate \( \alpha \). As the solution \( u(t, \alpha) \), \( v(t, \alpha) \) of this transformed equations system (the solution \( u(t, \alpha) \), \( v(t, \alpha) \) on the Lagrangian coordinate \( \alpha \)), we can define the solution \( u(t, \alpha) \), \( v(t, \alpha) \) of the limit equation (2) of MPS scheme.

Taking the implicit time derivative of the equation of continuity,

\[
-1 \frac{\rho(t + Dt, \alpha) - \rho(t, \alpha)}{\rho_0} = \sum_{i=1}^{N} \frac{\partial u_i(t + Dt, \alpha)}{\partial u_i(t, \alpha)} \tag{6}
\]

Taking implicit time derivative of the equation (2), we have

\[
\frac{v(t + Dt, \alpha) - v(t, \alpha)}{Dt} = \frac{\mu}{\rho_0} \sum_{i=1}^{N} \frac{\partial^2 v(t + Dt, \alpha)}{\partial u_i(t + Dt, \alpha)^2} - \frac{1}{\rho_0} \left( \frac{\partial P(t + Dt, \alpha)}{\partial u_i(t + Dt, \alpha)} \right) \delta_1 \delta_2 \cdots \delta_N + f(t, \alpha) \tag{7}
\]

By the above two equations (6) (7),

\[
\frac{\rho(t + Dt, \alpha) - \rho(t, \alpha)}{(Dt)^2} = \frac{\partial^2 P(t + Dt, \alpha)}{\partial u_i(t + Dt, \alpha)^2} \tag{8}
\]

Since the fluid flow is assumed to be incompressible, \( \rho(t + Dt, \alpha) \) must be controlled to the constant value \( \rho_0 \) in the left side of this equation (8). To do so, \( P(t + Dt, \alpha) \) must be computed as a solution of the following partial differential equation

\[
\frac{\rho_0 - \rho(t, \alpha)}{(Dt)^2} = \frac{\partial^2 P(t + Dt, \alpha)}{\partial u_i(t + Dt, \alpha)^2} \tag{9}
\]

By applying the pressure \( P(t + Dt, \alpha) \) which is a solution of the equation (9) to the limit equation (2) of MPS scheme, the position \( u(t, \alpha) \) and the velocity \( v(t, \alpha) \) are determined for any time \( t \geq 0 \) and any fluid particles \( \alpha \).

References